The Guided Moments formalism:
a brief introduction

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Based on Manuel R. Izquierdo et al.
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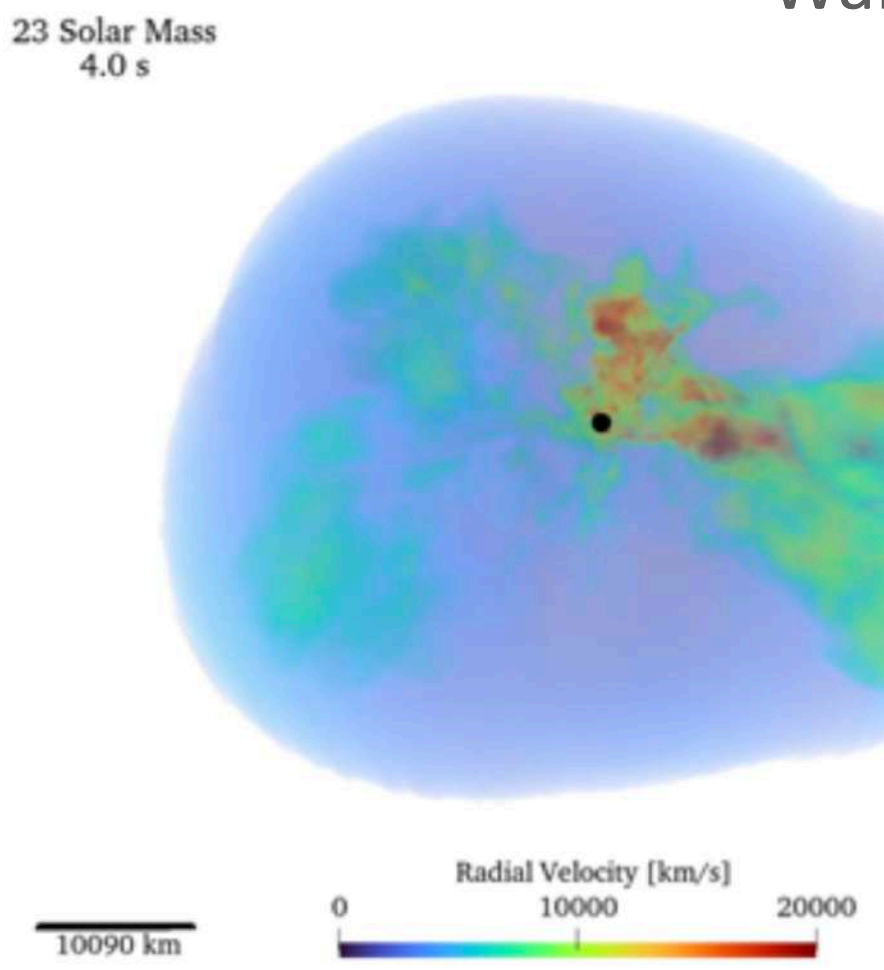
Slides adapted from Manuel's talk

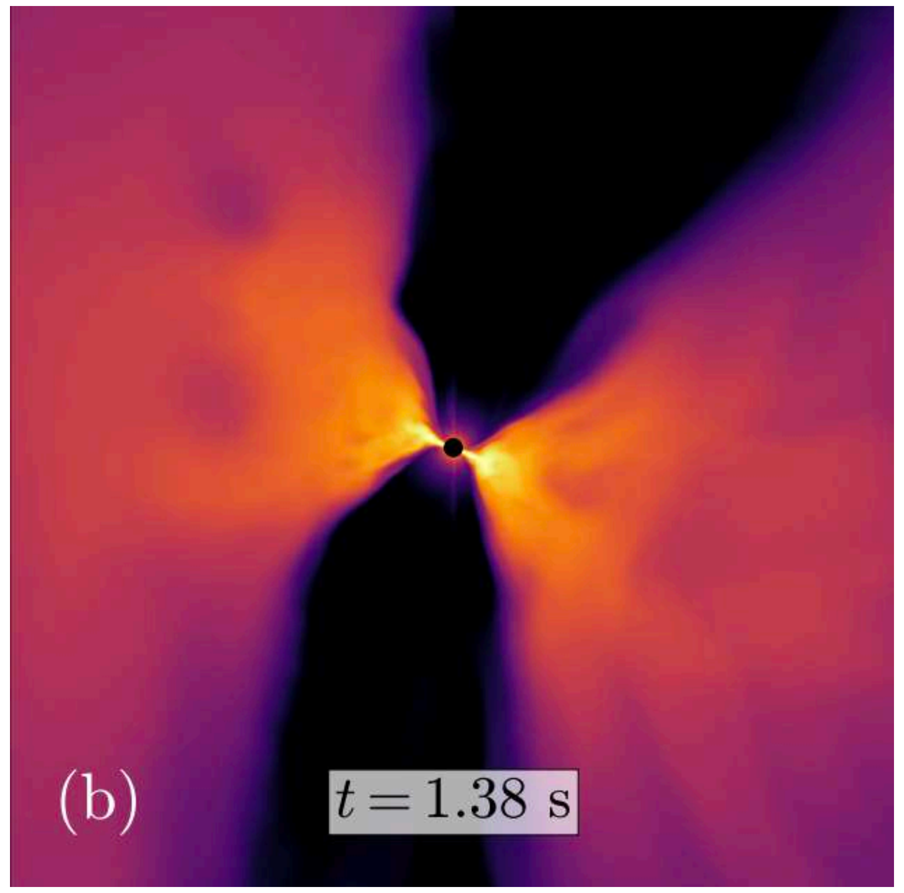
Why do we care about neutrinos?

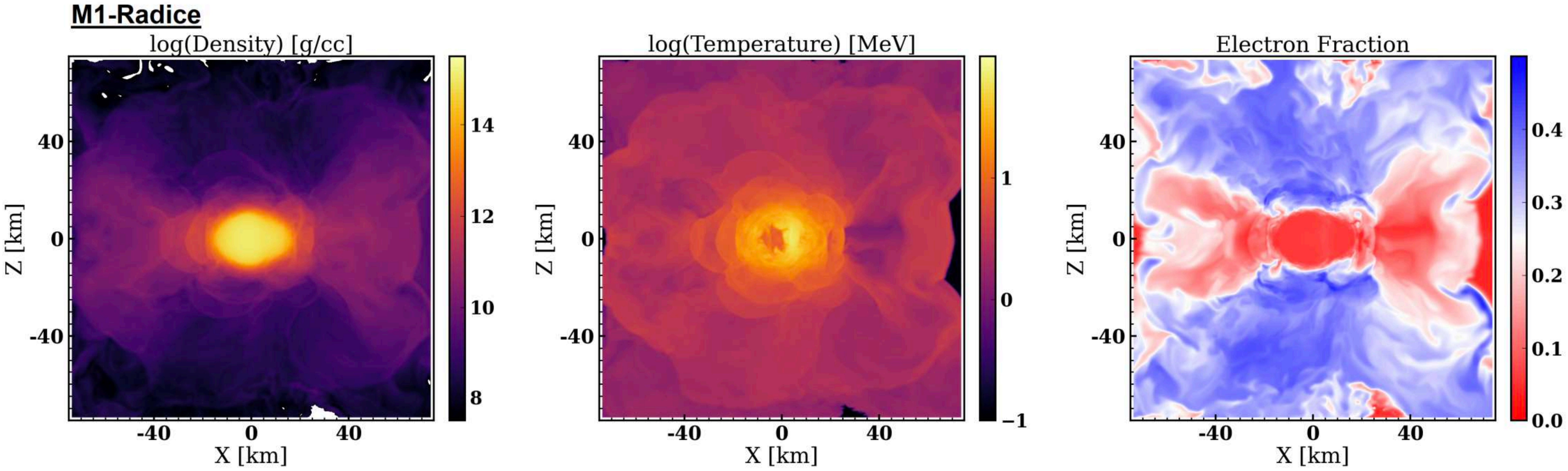
- BNS/BHNS mergers: r-process nucleosynthesis, matter outflows/winds
- Cooling of collapsar disks
- Core-collapse supernovae

Wang+23



Issa+25





Foucart+24

Background equations

- Einstein's field equations with stress-energy tensor comprising fluid and radiation contributions

$$G_{ab} = 8\pi T_{ab} = 8\pi \left(T_{ab}^{fl} + T_{ab}^{rad} \right)$$

- Radiation stress-energy tensor: considers three neutrino species

$$T_{ab}^{rad} = \sum_{\nu} T_{ab}^{\nu} = T_{ab}^{\nu_e} + T_{ab}^{\bar{\nu}_e} + T_{ab}^{\nu_x}$$

- GRRMHD equations:

$$\nabla_b T_{fl}^{ab} = -\sum_{\nu} S_{\nu}^{a}$$

Conservation equations of energy and momentum

$$\nabla_b^{\star} F^{ab} = 0$$

Maxwell's equations

$$\nabla_b (m_b (n_p + n_n) u^b) = 0$$

Baryon number conservation

The Boltzmann equation

- Distribution function of neutrinos in 7D phase-space: $f_\nu(x^a, p^a)$
- Number of neutrinos in 6D volume of phase-space: $N(t) = \int dx^3 \frac{dp^3}{h^3} f_\nu(t, x^i, p_j)$
- Evolution of the distribution function via the Boltzmann equation:

$$p^a \left[\frac{\partial f_\nu}{\partial x^a} - \Gamma^b_{ac} p^c \frac{\partial f_\nu}{\partial p^b} \right] = \left[\frac{\partial f_\nu}{\partial \tau} \right]_{coll}$$

Time evolution of a 6D function: an expensive computational challenge!

Approximate methods:

- Leakage schemes [Ruffert+97, Rosswog+03, ..]
- Truncated moment schemes (M0, M1, ..) [Thorne+97, Shibata+11, Radice+22]

Exact methods:

- Monte-Carlo algorithms [Foucart+22, Kawaguchi+23]
- Guided Moments [Izquierdo+24]

Truncated moments: M1 scheme

- Evolution of first two moments of the distribution function, i.e. energy density and flux density
- Grey-approximation: evolves energy integrated moments
- Stress-energy tensor using lower order moments:

$$T^{ab}_{rad} = Ju^a u^b + H^a u^b + H^b u^a + Q^{ab} \qquad \text{Co-moving frame}$$

$$T^{ab}_{rad} = En^a n^b + F^a n^b + F^b n^a + P^{ab} \qquad \text{Eulerian frame}$$

- Conservation equations in 3+1 form:

$$\partial_t(\sqrt{\gamma}E) + \partial_i\left[\sqrt{\gamma}(\alpha F^i - \beta^i E)\right] = \alpha\sqrt{\gamma}\left[P^{ij}K_{ij} - F^i(\partial_i\alpha)/\alpha - \mathcal{S}^a n_a\right]$$

$$\partial_t(\sqrt{\gamma}F_i) + \partial_j\left[\sqrt{\gamma}(\alpha P^j{}_i - \beta^j F_i)\right] = \sqrt{\gamma}\left[-E\partial_i\alpha + F_j\partial_i\beta^j + \frac{\alpha}{2}P^{kj}\partial_i\gamma_{kj} + \alpha\mathcal{S}^a\gamma_{ia}\right]$$

- Fluid-neutrino interaction term in 3+1 form:

$$\mathcal{S}_n = -\mathcal{S}^a n_a = W\left[(\eta + \kappa_s J) - (\kappa_a + \kappa_s)(E - F_i v^i)\right] \; ,$$

$$\mathcal{S}_i = \mathcal{S}^a\gamma_{ia} = W(\eta - \kappa_a J)v_i - (\kappa_a + \kappa_s)H_i$$

Truncated moments: M1 scheme

- Evolution equations not closed due to unknown second moment
- Closure relations: analytical prescription to interpolate the second moment between optically thick and thin regimes

$$P_{ij}^{\mathtt{M1}} = \frac{3\chi(\xi) - 1}{2} P_{ij}^{\mathrm{thin}} + \frac{3\,[1 - \chi(\xi)]}{2} P_{ij}^{\mathrm{thick}}$$

- Optically thick (diffusion) limit:

$$Q_{ab}^{\mathrm{thick}} = J_{\mathrm{thick}} h_{ab}/3$$

neutrinos and matter in thermodynamical equilibrium

- Optically thin limit:

$$P_{ij}^{\mathrm{thin}} = \frac{F_i F_j}{F^k F_k} E$$

free streaming neutrinos

- Minerbo Eddington factor: gives exact results in both limits

$$\chi(\xi) = \frac{1}{3} + \xi^2 \left(\frac{6 - 2\xi + 6\xi^2}{15} \right)$$

$$\xi \equiv \sqrt{\frac{H_a H^a}{J^2}}$$

Truncated moments: M1 scheme

- Stiff source terms appear due to neutrino-matter interactions:

$$\partial_t U + \partial_i F^i(U) = G(U) + \frac{1}{\epsilon} S(U)$$

$\epsilon \rightarrow \infty \Rightarrow$ system primarily follows hyperbolic evolution (explicit RK)

$\epsilon \rightarrow 0 \Rightarrow$ stiff term dominates; small perturbations in U leads to rapid changes (implicit RK)

- Full explicit and implicit solvers are computationally very expensive!

- Semi-implicit schemes: advection terms, metric sources (explicit); coupling with matter (implicit)

$$\text{(i)} \quad \frac{\boldsymbol{U}^* - \boldsymbol{U}^{(k)}}{\Delta t/2} = -\partial_i \boldsymbol{F}^i[\boldsymbol{U}^{(k)}] + \boldsymbol{G}[\boldsymbol{U}^{(k)}] + \boldsymbol{S}[\boldsymbol{U}^*],$$

$$\text{(ii)} \quad \frac{\boldsymbol{U}^{(k+1)} - \boldsymbol{U}^{(k)}}{\Delta t} = -\partial_i \boldsymbol{F}^i[\boldsymbol{U}^*] + \boldsymbol{G}[\boldsymbol{U}^*] + \boldsymbol{S}[\boldsymbol{U}^{(k+1)}].$$

[Radice+22]

Only 1st order accurate..

- Implicit-Explicit RK (IMEX): same CFL as RK4 + 2nd order accurate [Izquierdo+23]

Monte-Carlo algorithm

- Traces the paths of an ensemble of neutrino packets N_T, each containing N_k neutrinos

$$f_\nu \sim f_{(\nu)} = \sum_{k=1}^{N_\mathrm{T}} N_k \delta^3(x^i - x^i_k)\delta^3(p_i - p^k_i)$$

- Each packet carries information on: $\left(t, x^i, p_i, N_k, \nu_a, E_{\nu_a}, \eta, \kappa_a, \kappa_s, ..\right)$

- Radiation stress-energy tensor: $\bar{T}^{ab}(t, x^i) = \sum_{k\in\Delta V} N_k \frac{p^a_k p^b_k}{\sqrt{-g}\Delta V p^t_k}$

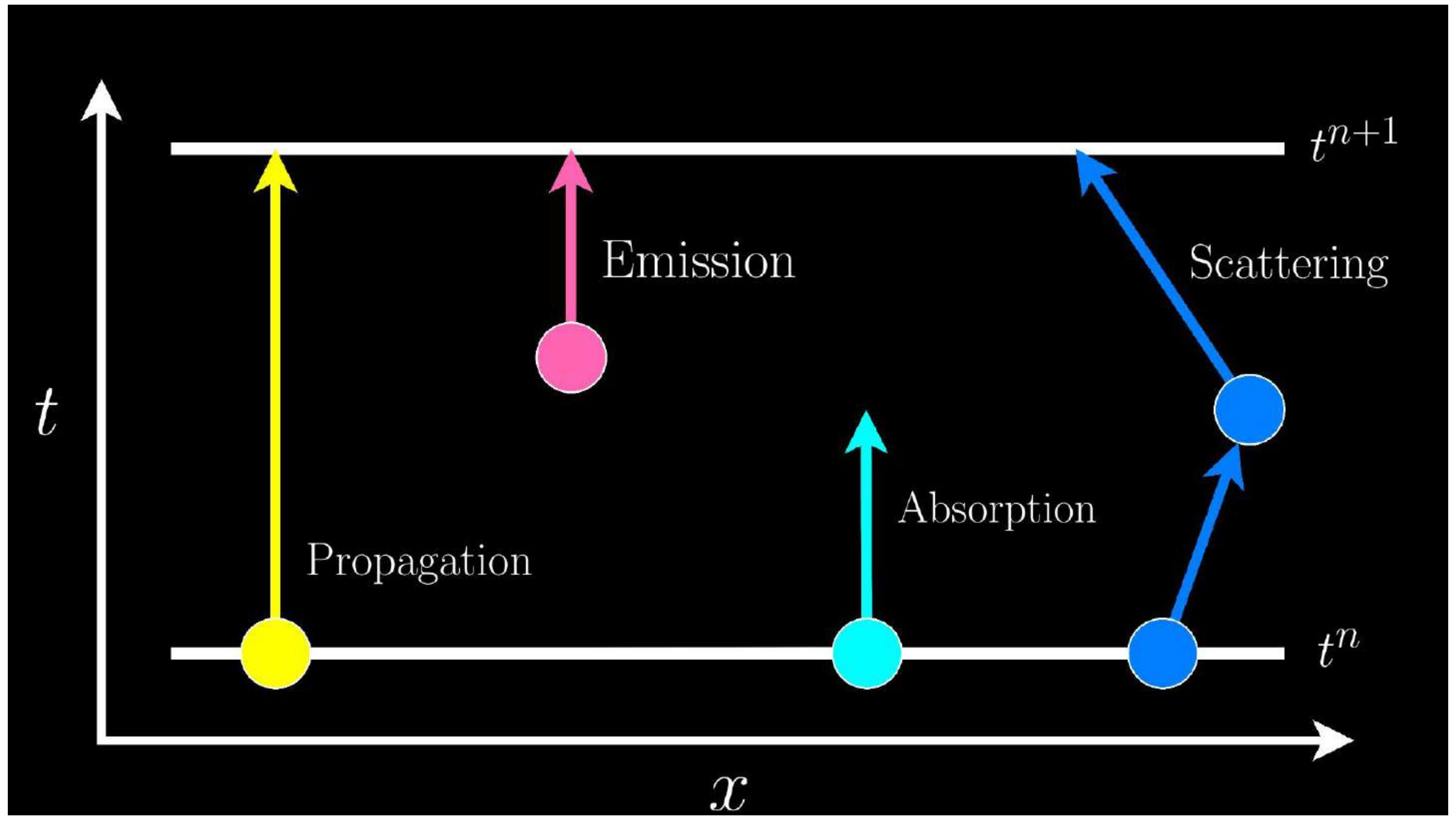
- MC moments:

$$\bar{E} = \sum_{k\in\Delta V} \frac{N_k \epsilon_k}{\sqrt{\gamma}\Delta V} \qquad \bar{F}_a = \sum_{k\in\Delta V} \frac{N_k \epsilon_k l^k_a}{\sqrt{\gamma}\Delta V} \qquad P^{\mathtt{MC}}_{ij} = \bar{P}_{ij} = \sum_{k\in\Delta V} N_k \frac{p^k_i p^k_j}{\sqrt{-g}\Delta V p^t_k}$$

where $\epsilon_k = -p^k_a n^a$ is the neutrino energy measured in Eulerian frame, and $n^a l^k_a = 0$

Monte-Carlo algorithm

- Step 1: neutrino packets are created with following information $(t, x^i, p_i, N_k, \nu_a, E_{\nu_a}, \eta, \kappa_a, \kappa_s, ..)$
- Step 2: prescriptions for free streaming (propagation)
  - Neutrinos move along geodesics until they reach the domain boundary or undergo interactions
- Step 3: prescriptions for emission/absorption/scattering (probabilistic)
  - Emission results in creation of new neutrino packets
  - Based on optical depth along the geodesic, packets are either removed due to absorption or have momenta updated considering only elastic scattering
- Step 4: MC moments are updated from the new positions and momenta



M1 versus MC

M1 formalism

- Advantages
  - Conservative 3+1 system of equations
  - Grid based method
- Disadvantages
  - Approximate formalism
  - Non physical behavior in optically thin regions
  - Energy integrated moments considered

MC formalism

- Advantages
  - Converges to exact solution
  - Energy-dependent scheme
- Disadvantages
  - Expensive in optically thick regimes
  - Statistical errors decrease as the square root of the number of samples: $O(N^{-1/2})$

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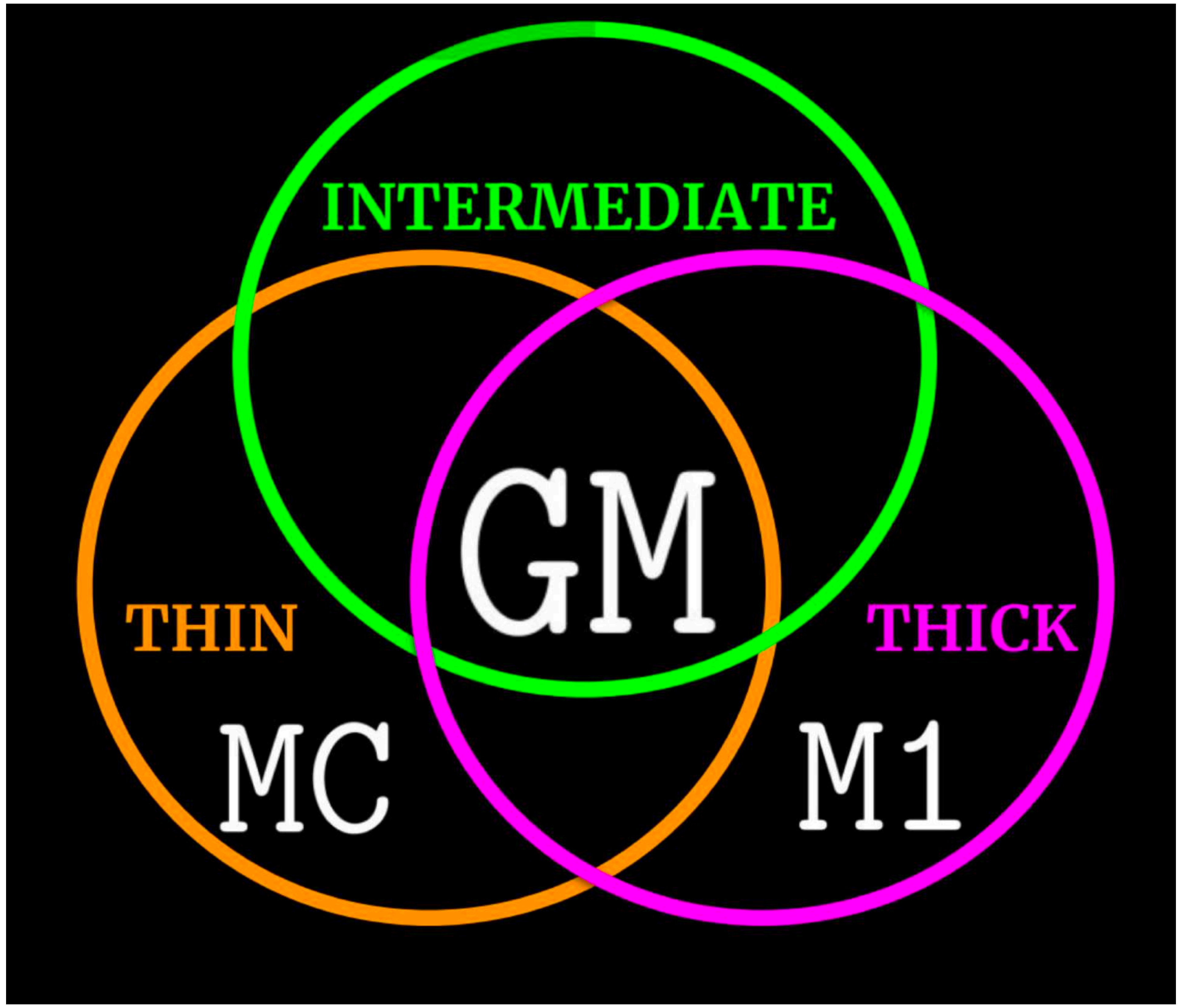
Guided Moments formalism

- Hybrid method coupling M1 and MC schemes
- Inspired by [Degond+11, Foucart18]

- Advantages
  - MC informed closure relation for M1, accurate for opt. thin regimes
  - Avoids expensive cost of MC in opt. thick regimes
  - Converges to exact solution (in both regimes)

- Disdvantages
  - Computationally more expensive than M1
  - Transition functions to be heuristically determined



Guided Moments formalism

1) MC informed closure for M1

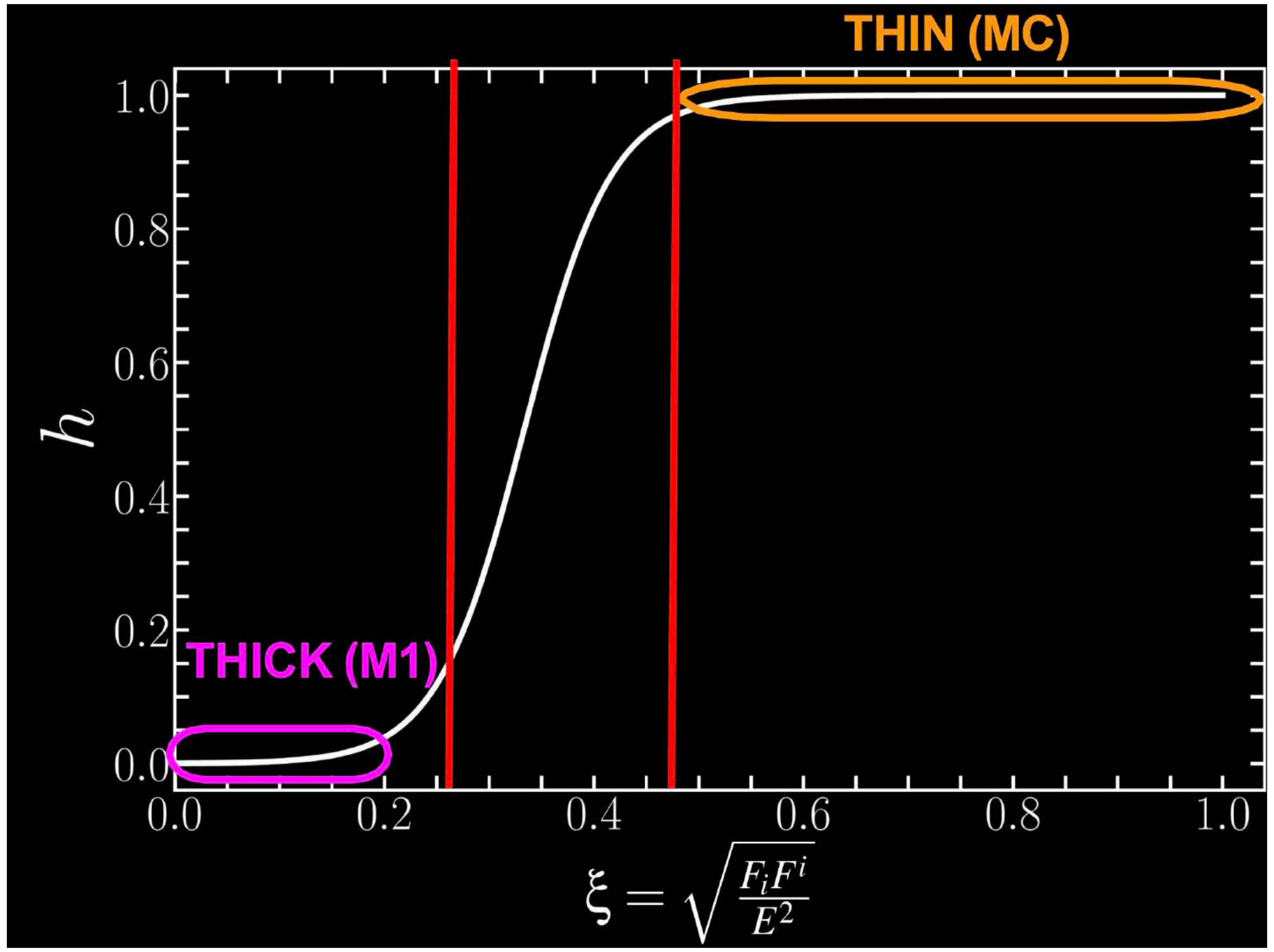
$$P_{ij}^{\mathtt{M1}} = \frac{3\chi(\xi)-1}{2} P_{ij}^{\mathrm{thin}} + \frac{3\left[1-\chi(\xi)\right]}{2} P_{ij}^{\mathrm{thick}}$$

$$P_{ij}^{\mathtt{MC}} = \bar{P}_{ij} = \sum_{k\in\Delta V} N_k \frac{p_i^k p_j^k}{\sqrt{-g}\Delta V p_k^t}$$

- GM second moment: $P_{ij}^{\mathtt{GM}} = h(\xi) P_{ij}^{\mathtt{MC}} + \left[1-h(\xi)\right] P_{ij}^{\mathtt{M1}}$

- Transition function: $h(\xi) = \dfrac{1}{1+e^{-2\,k(\xi-\xi_0)}}$

- Solution: substitute $P_{ij}^{\mathtt{M1}} \to P_{ij}^{\mathtt{GM}}$ (GM1 solution)



Guided Moments formalism

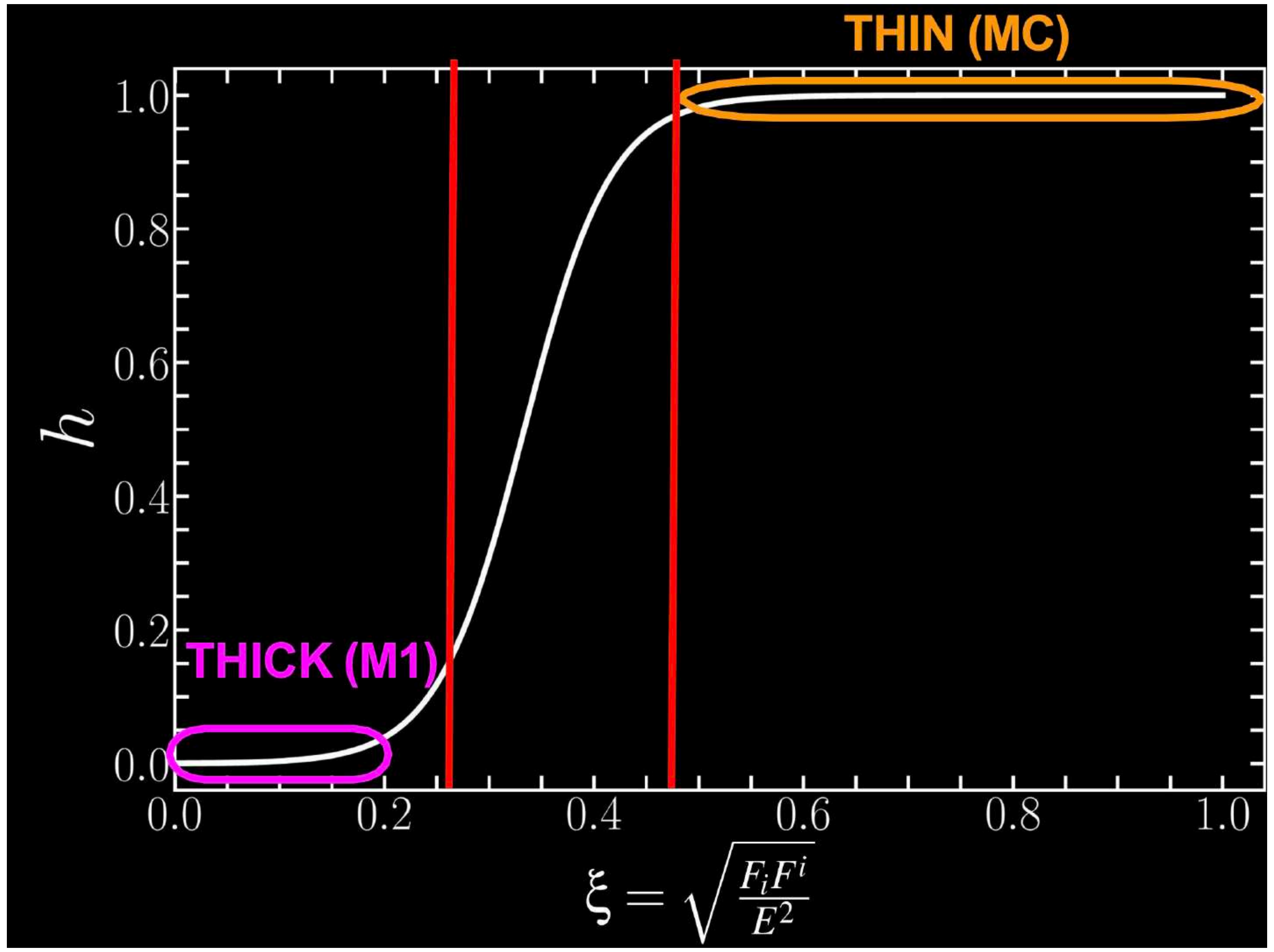
2) Matching of lowest moments

$$\mathcal{J}_a^{\mathtt{M1}} = \mathcal{J}_a \equiv -T_{ab}n^b = En_a + F_a$$

$$\mathcal{J}_a^{\mathtt{MC}} = \bar{\mathcal{J}}_a \equiv -\bar{T}_{ab}n^b = \sum_{k\in\Delta V} N_k \frac{p_a^k}{\sqrt{\gamma}\Delta V}$$

- GM first moment: $\mathcal{J}_a^{\mathtt{GM}} = h(\xi)\mathcal{J}_a^{\mathtt{MC}} + [1-h(\xi)]\mathcal{J}_a^{\mathtt{M1}}$
- Transition function: $h(\xi) = \dfrac{1}{1+e^{-2\,k(\xi-\xi_0)}}$
- Solution: substitute $\mathcal{J}_a^{\mathtt{MC}} \to \mathcal{J}_a^{\mathtt{GM}}$ (GMC solution)

This substitution involves rescaling (via Lorentz boost transformations) the neutrino 4-momentum of MC using GM solution



Algorithm of the Guided Moments (GM)

Manuel's slide

for each substep **do**

  Evolve M1 substep

  Evolve MC substep

  Compute the lowest moments of the GM $\rightarrow$

$$\mathcal{J}_a^{\mathrm{GM}} = h(\xi)\mathcal{J}_a^{\mathrm{MC}} + [1 - h(\xi)]\mathcal{J}_a^{\mathrm{M1}}$$

  Compute transformation matrices $\rightarrow$

$$\Lambda_b^a = \Lambda_b^a(\mathcal{J}^{\mathrm{GM}}, \mathcal{J}^{\mathrm{MC}})$$

  Transform MC 4-momentums to match $\mathcal{J}_a^{\mathrm{GM}}$ $\rightarrow$

$$p_b^{\mathrm{GM},k} = \frac{\mathcal{J}^{\mathrm{MC}}}{\mathcal{J}^{\mathrm{GM}}} p_k^{\mathrm{MC},a} \Lambda_a^b$$

  Update MC 4-momentums and compute new P_{ij}^{MC}

  Compute GM pressure tensor $\rightarrow$

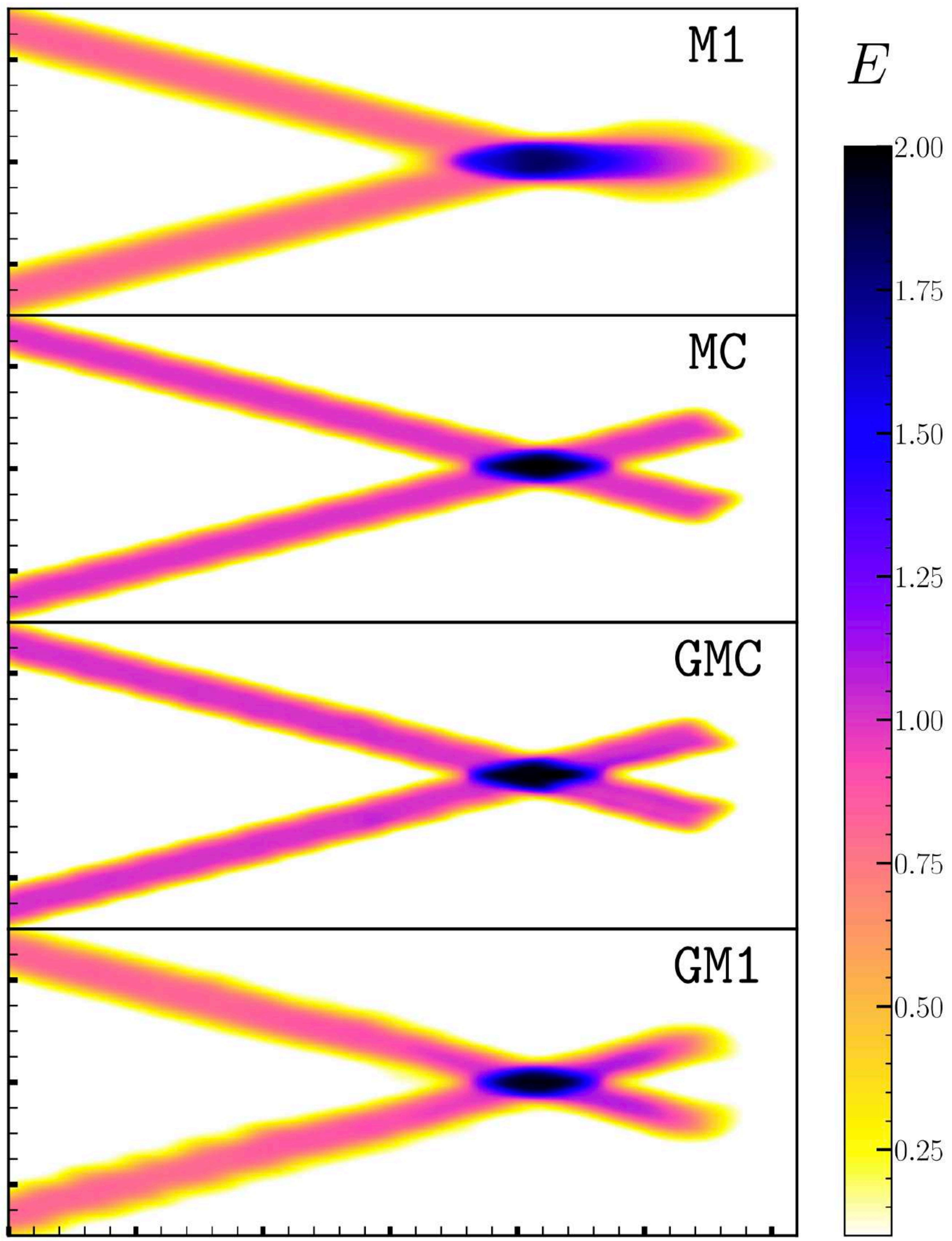
$$P_{ij}^{\mathrm{GM}} = h(\xi)P_{ij}^{\mathrm{MC}} + [1 - h(\xi)]P_{ij}^{\mathrm{M1}}$$

  Substitute $P_{ij}^{\mathrm{M1}} \rightarrow P_{ij}^{\mathrm{GM}}$

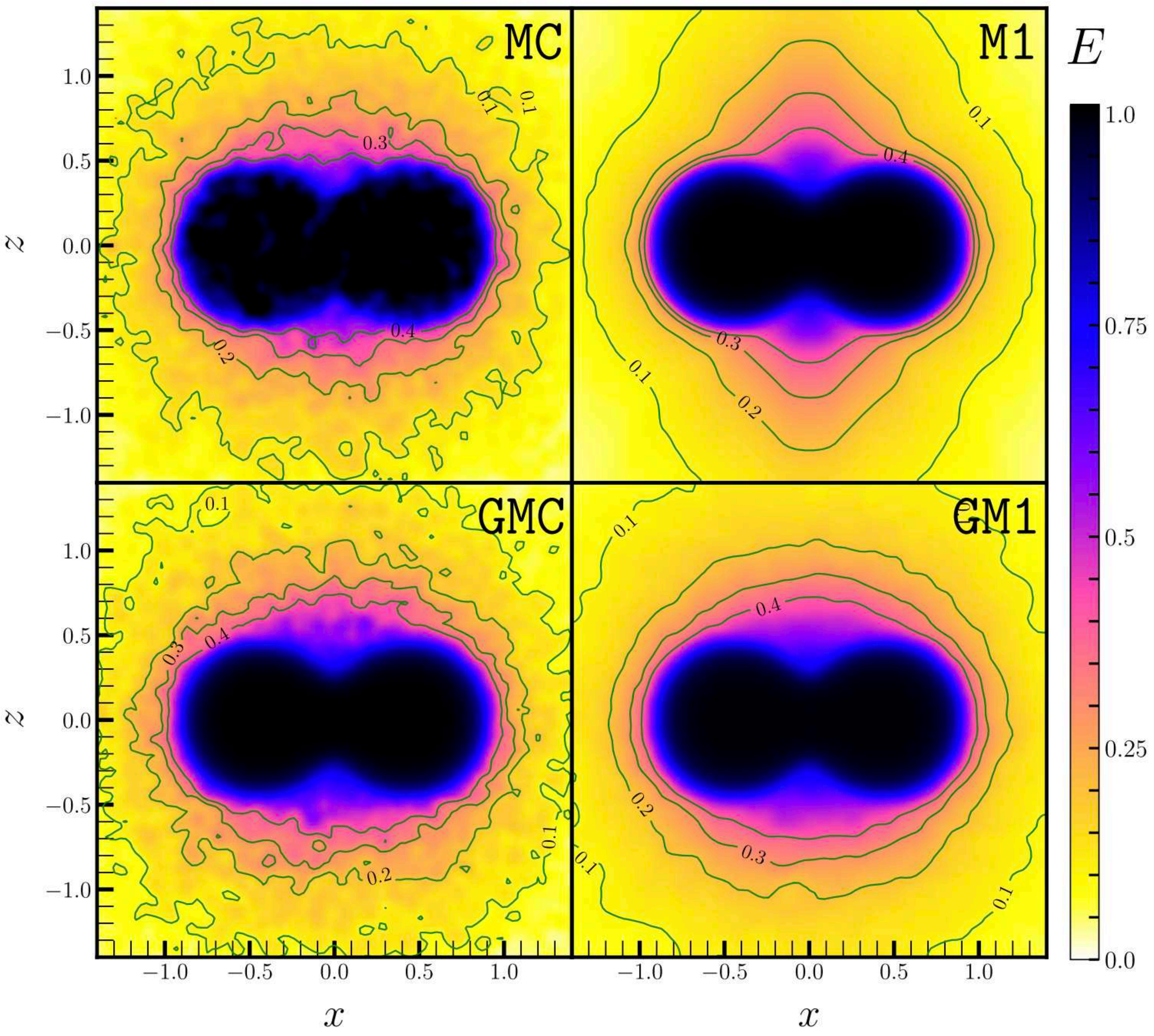
end for

Numerical tests

Double beam test



Radiating/absorbing Torus



Back-up slides

Algorithm of Monte-Carlo (MC) [simplified version]

Manuel's slide

Initialization: $N_{\text{packets}}, E_p, \Delta x, \Delta t, \nu, \kappa_a, \kappa_s, \eta, u^\mu ...$
for $j = 1$ to n_{iters} **do**
  Compute $N_p = \frac{E_{\text{ems}}}{E_{\text{packet}}} \sim \Delta t \kappa_a N_{\text{tr}}$
  Emission: Generate packets
  for $i = 1$ to N_{packets} **do**
    $\Delta t_\star = \Delta t$
    while $\Delta t_\star \leq \Delta t$ **do**
      $\Delta t_{a,s} = -\ln(r_{a,s}) \frac{p^t}{\kappa_{a,s}\nu}$ where $r_{a,s} \in [10^{-70}, 1]$
      $\Delta t_{\text{min}} = \text{min}(\Delta t_*, \Delta t_a, \Delta t_s)$
      if $\Delta t_{\text{min}} == \Delta t_\star$ **then**
        Propagation: Propagate Δt_*
      else if $\Delta t_{\text{min}} == \Delta t_a$ **then**
        Absorption: Delete packet
      else if $\Delta t_{\text{min}} == \Delta t_s$ **then**
        Scattering: Propagate Δt_s and modify p^μ
        $\Delta t_* = \Delta t - \Delta t_s$
      end if
    end while
  end for
  Reconstruct moments $\equiv (x^i, p_i) \rightarrow (E, F_i, P_{ij})$ **[Time consuming!]**
end for